Chromatic Polynomials for $J(\prod H)I$ Strip Graphs and their Asymptotic Limits

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Abstract

We calculate the chromatic polynomials P for n-vertex strip graphs of the form $J(\prod_{\ell=1}^{m} H)I$, where J and I are various subgraphs on the left and right ends of the strip, whose bulk is comprised of m-fold repetitions of a subgraph H. The strips have free boundary conditions in the longitudinal direction and free or periodic boundary conditions in the transverse direction. This extends our earlier calculations for strip graphs of the form $(\prod_{\ell=1}^{m} H)I$. We use a generating function method. From these results we compute the asymptotic limiting function $W = \lim_{n\to\infty} P^{1/n}$; for $q \in \mathbb{Z}_+$ this has physical significance as the ground state degeneracy per site (exponent of the ground state entropy) of the q-state Potts antiferromagnet on the given strip. In the complex q plane, W is an analytic function except on a certain continuous locus $\mathcal{B}$. In contrast to the $(\prod_{\ell=1}^{m} H)I$ strip graphs, where $\mathcal{B}$ (i) is independent of I, and (ii) consists of arcs and possible line segments that do not enclose any regions in the q plane, we find that for some $J(\prod_{\ell=1}^{m} H)I$ strip graphs, $\mathcal{B}$ (i) does depend on I and J, and (ii) can enclose regions in the q plane. Our study elucidates the effects of different end subgraphs I and J and of boundary conditions on the infinite-length limit of the strip graphs.

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I. INTRODUCTION

In this paper we generalize our previous study [1] of chromatic polynomials and their asymptotic limits on strip graphs. The motivation for this work is that there are substances that exhibit nonzero ground state entropy, such as ice [2]. A simple model that exhibits ground state entropy is the q-state Potts antiferromagnet (PAF) [3,4] for sufficiently large values of q. The ground state entropy is related here to the ground state degeneracy per site W by $S_0 = k_B \ln W$. There is a direct connection with mathematical graph theory via the elementary equality $Z(G, q, T = 0)_{PAF} = P(G, q)$, where $Z(G, q, T = 0)_{PAF}$ is the partition function of the zero-temperature q-state Potts AF and $P(G, q)$ is the chromatic polynomial [5,6], giving the number of ways of coloring the n-vertex graph G with q colors such that no adjacent vertices have the same color, and the consequent equality

$$W(\{G\}, q) = \lim_{n \to \infty} P(G, q)^{1/n} \tag{1.1}$$

where $\{G\}$ denotes the $n \to \infty$ limit of the family of graphs G (e.g., a regular lattice, $\{G\} = \Lambda$ with some specified boundary conditions). Since $P(G, q)$ is a polynomial, there is a natural generalization of q from $q \in \mathbb{Z}_+$ to $q \in \mathbb{R}_+$, and indeed to $q \in \mathbb{C}$. In general, $W(\{G\}, q)$ is an analytic function in the q plane except along a certain continuous locus of points, which we denote $\mathcal{B}$ (and at possible isolated singularities which will not be important here). In the limit as $n \to \infty$, the locus $\mathcal{B}$ forms by means of a coalescence of a subset of the zeros of $P(G, q)$ (called chromatic zeros of G) [7]- [10].

In a series of papers [11]- [16] two of us have calculated and analyzed $W(\{G\}, q)$, both for physical values of q (via rigorous upper and lower bounds, large–q series calculations, and Monte Carlo measurements) and for the generalization to complex values of q. In Ref. [1] we presented exact calculations of chromatic polynomials and the asymptotic limiting W functions for strip graphs of a variety of regular lattices. For this purpose we developed and

applied a generating function method. Specifically, we considered strip graphs of regular lattices of type s of the form

$$(G_s)_{m;I} = (\prod_{\ell=1}^{m} H)I \tag{1.2}$$

where s stands for square, triangular, honeycomb, etc. Thus, such a graph is composed of m repetitions of a subgraph H attached to an initial subgraph I on one end, which, by convention, we take to be the right end. In the case of homogeneous strips, $I = H$. As is implicit in the above notation, we picture such strip graphs as having a length of L_x vertices in the horizontal (x) direction and a width of L_y vertices in the transverse (vertical, y) direction. Since for the limit of infinite length strips, the physical ground state degeneracy per site, W, and ground state entropy, S_0, are independent of the boundary conditions used in the longitudinal direction, we used free boundary conditions for technical simplicity in Ref. [1]. (Homeomorphic expansions of these strip graphs were also studied in Ref. [17].)

In the present work we generalize our previous study by considering strip graphs of regular lattices in which there may be special subgraphs on both the left and right ends of the strip:

$$(G_s)_{m,J,I} = J(\prod_{\ell=1}^{m} H)I \tag{1.3}$$

The end subsgraphs I and J may be different from each other and from the vertical rungs of the strip (thought of as a horizontal ladder). As before, we use the simplest type of boundary conditions – free ones – in the longitudinal direction, since in the limit of interest, namely that of infinite length, the physical ground state entropy is independent of these boundary conditions. However, since the size of the limit is finite in the transverse direction, it is of interest to consider both free and periodic boundary conditions in this transverse direction, and we do that here. Again, we use the generating function method. For the infinite-length limit of $(\prod_{\ell=1}^{m} H)I$ strip graphs, we proved that the nonanalytic locus $\mathcal{B}$ is independent of I, which is plausible, since in this limit, the subgraph I occupies a vanishingly

small part of the full strip graph [1]. However, our study of the infinite-length limit of $J(\prod_{\ell=1}^{m} H)I$ strip graphs reveals that in some cases, $\mathcal{B}$ does depend on the end subgraphs I and J. Furthermore, whereas for $(\prod_{\ell=1}^{m} H)I$ strip graphs we found that $\mathcal{B}$ consists of arcs and possible line segments that do not enclose any regions in the q plane, in contrast, for $J(\prod_{\ell=1}^{m} H)I$ strip graphs, we show here that in certain cases $\mathcal{B}$ does enclose regions in the q plane. See also Ref. [18] and a companion paper on strip graphs of a particular Archimedean lattice [19] denoted as $(3^3 \cdot 4^2)$.

It will be convenient to define the function

$$D_k(q) = \frac{P(C_k, q)}{q(q-1)} = \sum_{s=0}^{k-2} (-1)^s \binom{k-1}{s} q^{k-2-s} \tag{1.4}$$

where $P(C_k, q)$ is the chromatic polynomial for the circuit (cyclic) graph C_k with k vertices,

$$P(C_k, q) = (q-1)^k + (-1)^k (q-1) \tag{1.5}$$

This paper is organized as follows. In Section 2 we describe our present implementation of our generating function method. In sections 3-5 we apply this method to strip graphs of the square, honeycomb and triangular lattices. In section 6 we study strips of a certain Archimedean lattice known as the $(4 \cdot 8^2)$ lattice. Sections 7 and 8 present results for the square and triangular lattices with periodic boundary conditions in the transverse direction (equivalently, these may be called cylindrical boundary conditions). Some further discussion and conclusions are given in section 9.

II. GENERATING FUNCTION METHOD AND END SUBGRAPHS

We denote the chromatic polynomial describing the coloring of the strip graph $(G_s)_{m,J,I}$ with q colors as $P((G_s)_{m,J,I}, q)$. In the generating function method, this chromatic polyno-

mial is given by means of an expansion, in terms of an auxiliary variable x, of a generating function $\Gamma(G_{s,J,I}, q, x)$:

$$\Gamma(G_{s,J,I}, q, x) = \sum_{m=0}^{\infty} P((G_s)_{m,J,I}, q)x^m \tag{2.1}$$

As before for strip graphs of the form (1.2), we find that the generating function $\Gamma(G_{s,J,I}, q, x)$ is a rational function of the form

$$\Gamma(G_{s,J,I}, q, x) = \frac{\mathcal{N}(G_{s,J,I}, q, x)}{\mathcal{D}(G_{s,J,I}, q, x)} \tag{2.2}$$

with (suppressing J, I dependence in the notation)

$$\mathcal{N}(G_s, q, x) = \sum_{j=0}^{j_{max}} a_{G_s,j}(q)x^j \tag{2.3}$$

and

$$\mathcal{D}(G_s, q, x) = 1 + \sum_{k=1}^{k_{max}} b_{G_s,k}(q)x^k \tag{2.4}$$

where the $a_{G_s,i}$ and $b_{G_s,i}$ are polynomials in q. The degrees j_{max} and k_{max} of $\mathcal{N}(G_s, q, x)$ and $\mathcal{D}(G_s, q, x)$ as polynomials in x depend on the type and width of the strip G_s.

The method that we shall use for calculating the generating function is based on the addition-contraction theorem from graph theory. This was briefly described in Ref. [1], together with the equivalent deletion-contraction theorem; here we shall give a more detailed discussion. We first recall the statement of the addition-contraction theorem: let G be a graph, and let v and v' be two non-adjacent vertices in G. Form (i) the graph $G_{add.}$ in which one adds a bond connecting v and v', and (ii) the graph $G_{contr.}$ in which one identifies v and v'. Then the chromatic polynomial for G is equal to the sum of the chromatic polynomial for the graphs $G_{add.}$ and $G_{contr.}$. For our discussion, we may begin by taking J and I to be the same as the other vertical rungs of the strip. By applying the addition-contraction theorem to the right-hand side of the initial H subgraph in $(G_s)_m$, we obtain a set of strip graphs

$(G'_{s,j})_m$ with complete [20] subgraphs labeled by j on the right-hand end. For example, let us consider a strip $(G_s)_m = \prod_{\ell=1}^m H$ where the width is such that to go from one transverse end to the other one traverses at least four vertices (including the two vertices on the transverse boundaries), and label the four vertices on the right-hand longitudinal end, in sequence, as v_1, v_2, v_3, v_4. (For a strip of the square lattice with free transverse boundary conditions, this example corresponds to a strip of width $L_y = 4$. For other types of strip this may correspond to different values of L_y, as illustrated by explicit calculations below.) Now apply the addition-contraction theorem to the pair v_1, v_4; this yields the equation

$$P((G'_s)_{m-1}, q) = P((G'_{s,b(1,4)})_m, q) + P((G'_{s,v_1=v_4})_m, q) \tag{2.5}$$

where $(G'_{s,b(1,4)})_m$ denotes the strip graph with the right-hand end modified by the addition of the bond connecting vertices v_1 and v_4, and $(G'_{s,v_1=v_4})_m$ denotes the strip graph with the right-end modified by identifying vertices v_1 and v_4. Applying the theorem again to each of these two strip graphs, one obtains the equation

$$P((G'_s)_{m-1}, q) = P((G'_{s,K_4})_m, q) + P((G'_{s,K_3(v_1=v_3)})_m, q) + P((G'_{s,K_3(v_2=v_4)})_m, q)$$

$$+ P((G'_{s,K_2(v_1=v_3,v_2=v_4)})_m, q) + P((G'_{s,K_3(v_1=v_4)})_m, q) \tag{2.6}$$

Let us label the five resultant complete graphs on the right-hand end via the label i, with chromatic polynomials denoted by $P((G'_{s,i})_0, q)$. Thus,

$$P((G'_{s,1})_0, q) = P(K_4, q) = q(q-1)(q-2)(q-3) \tag{2.7}$$

$$P((G'_{s,i})_0, q) = P(K_3, q) = q(q-1)(q-2) \quad \text{for} \quad i = 2, 3, 5 \tag{2.8}$$

$$P((G'_{s,4})_0, q) = P(K_2, q) = q(q-1) \tag{2.9}$$

Next, working one's way leftward from the right-hand end, apply the addition-contraction theorem in a similar manner to the next set of transverse vertices, and label the five resultant complete graphs with the label j. Define a (square) matrix M (in this case 5×5) with

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elements consisting of the chromatic polynomials of this set of graphs with complete subgraphs i on the right and j on the left. This procedure transforms the initial strip into a sum of factorized strips with parts that overlap in complete graphs. Therefore the intersection theorem yields, for $m \geq 1$,

$$P((G'_{s,j})_m, q) = \sum_i \frac{M_{ji} P((G'_{s,i})_{m-1}, q)}{P((G'_{s,i})_0, q)} = \sum_i (MD)_{ji} P((G'_{s,i})_{m-1}, q), \qquad (2.10)$$

where D is a (5×5) diagonal matrix with elements

$$D_{i,i} = \frac{1}{P((G'_{s,i})_0, q)} \quad 1 \leq i \leq 5 \qquad (2.11)$$

with the ordering as given above. (Do not confuse this matrix D with the denominator $\mathcal{D}$ of the generating function Γ defined in eq. (2.4). Because of (2.6), the generating function of G'_s can be written as

$$\Gamma(G'_s, q, x) = \sum_j \Gamma(G'_{s,j}, q, x) \qquad (2.12)$$

Writing the generating function of $G'_{s,j}$ in the form

$$\Gamma(G'_{s,j}, q, x) = \sum_{m=1}^{\infty} P((G'_{s,j})_m, q) x^{m-1} \qquad (2.13)$$

and using (2.10) with (2.13), we obtain

$$\Gamma(G'_{s,j}, q, x) = P((G'_{s,j})_1, q) + \sum_i x \, (MD)_{ji} \sum_{m=2}^{\infty} P((G'_{s,i})_{m-1}, q) x^{m-2}$$

$$= P((G'_{s,j})_1, q) + \sum_i x \, (MD)_{ji} \Gamma(G'_{s,i}, q, x) \qquad (2.14)$$

The special case $m = 1$ in eq. (2.10) yields

$$P((G'_{s,j})_1, q) = \sum_i M_{ji} \qquad (2.15)$$

Let us define a vector $\boldsymbol{\Gamma}$ with elements $\Gamma(G'_{s,j}, q, x)$, where the index j also labels the elements of the vector. Thus, eq. (2.12) can be written as

$$\Gamma(G'_s, q, x) = \mathbf{v_b}^T \cdot \mathbf{\Gamma} \tag{2.16}$$

where $\mathbf{v_b}$ is a vector with all elements equal to unity. In the example described above, $\mathbf{v_b} = (1, 1, 1, 1, 1)^T$. (Do not confuse the vectors $\mathbf{v_b}$ with the vertices v_i.) Similarly, let us define the vector $\mathbf{P_1}$ with elements $P((G'_{s,j})_1, q)$. Then eq. (2.15) yields

$$\mathbf{P_1} = M\mathbf{v_a} \tag{2.17}$$

where in this case $\mathbf{v_a} = \mathbf{v_b}$. In vector form, eq. (2.14) becomes

$$\mathbf{\Gamma} = \mathbf{P_1} + xMD\mathbf{\Gamma} \tag{2.18}$$

Solving for $\mathbf{\Gamma}$, we get

$$\mathbf{\Gamma} = (1 - xMD)^{-1}\mathbf{P_1} \tag{2.19}$$

Henceforth, for notational simplicity, we drop the prime on G and indicate the end subgraphs I and J explicitly. Using (2.16) and (2.17) in eq. (2.19), we obtain

$$\Gamma(G_{s,J,I}, q, x) = \mathbf{v_b}^T(1 - xMD)^{-1}M\mathbf{v_a} = \mathbf{v_b}^T \sum_{m=0}^{\infty} x^m(MD)^m M\mathbf{v_a}, \tag{2.20}$$

where in this case, for the strip G_s, $\mathbf{v_a} = (1, 1, 1, 1, 1)^T$ and $\mathbf{v_b} = \mathbf{v_a}$. Note that in general $\mathbf{v_a}$ and $\mathbf{v_b}$ can be chosen independently, and different choices of these vectors correspond to different end graphs J and I, respectively. Thus, we can equivalently write (2.1) as

$$P((G_{s,J,I})_m, q) = \mathbf{v_b}^T(MD)^m M\mathbf{v_a} \tag{2.21}$$

Factorizing the denominator of the generating function (2.20), we have

$$\mathcal{D}(G_{s,J,I}, q, x) = \prod_r (1 - \lambda_r(q)x), \tag{2.22}$$

where the $\lambda_r(q)'s$ are eigenvalues of the product of matrices MD. The subgraphs on the two ends of the strip (defined by $\mathbf{v_a}$ and $\mathbf{v_b}$) determine which eigenvalues enter in the product

in equation (2.22). Using this method, we have calculated chromatic polynomials and their asymptotic limits for various strip graphs of the form $(G_{s,J,I})_m = J(\prod_{\ell=1}^m H)I$. We define $(G_{s,J,I})_{m=0}$ as being equal to JI, where the end graphs I and J are, in general, different. For technical convenience, we also use a slightly different labelling convention than in Ref. [1], which can be illustrated as follows: for strips of the square lattice, for example, we take the end graphs I and J to refer to the respective sets of vertical bonds on the right and left ends of the strip, whereas in Ref. [1], I referred to the first vertical layer of squares. We now present some of these calculations and the analytic structure of the corresponding $W(\{G_{sJ,I}\}, q)$.

We start by defining the vectors which will be used to describe the subgraphs on the ends of the strips. Let us denote the four vertices on one end, in sequence, as v_i, with $i = 1, ..., 4$ and consider end subgraphs to be described by vectors $\mathbf{v_a}$ and $\mathbf{v_b}$ chosen from a set of possibilities that we list below:

(i)

$$\mathbf{v_1} = (1, 0, 0, 0, 0)^T \tag{2.23}$$

which corresponds to a strip with a complete graph K_4 on the end, as shown in Fig. 1(a) for a strip of the square lattice of width $L_y = 4$. This K_4 is formed by adding edges between vertices v_1 and v_4, v_1 and v_3, and v_2 and v_3.

(ii)

$$\mathbf{v_2} = (0, 1, 0, 0, 0)^T \tag{2.24}$$

corresponding to a strip with vertices v_1 and v_3 contracted and an extra edge connecting vertices v_2 and v_4, yielding a $K_3(v_1 = v_3)$ on the end. Fig. 1(b) illustrates this for a strip of the square lattice of width $L_y = 4$.

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(iii)

$$\mathbf{v_3} = (0, 0, 1, 0, 0)^T \tag{2.25}$$

which corresponds to a strip with vertices v_2 and v_4 contracted and an extra edge connecting vertices v_1 and v_3, yielding a $K_3(v_2 = v_4)$ on the end. Fig. 1(c) illustrates this for a strip of the square lattice of width $L_y = 4$.

(iv)

$$\mathbf{v_4} = (0, 0, 0, 1, 0)^T \tag{2.26}$$

which corresponds to a strip with vertices v_1 contracted with v_3 and v_2 contracted with v_4, forming a $K_2(v_1 = v_3, v_2 = v_4)$ on the end. Fig. 1(d) illustrates this for a strip of the square lattice of width $L_y = 4$.

(v)

$$\mathbf{v_5} = (0, 0, 0, 0, 1)^T \tag{2.27}$$

which corresponds to a strip with vertices v_1 and v_4 contracted, yielding a $K_3(v_1 = v_4)$ on the end. Fig. 1(e) illustrates this for a strip of the square lattice of width $L_y = 4$.

(vi)

$$\mathbf{v_6} = (1, 1, 0, 0, 0)^T \tag{2.28}$$

which corresponds to a strip with one extra edge connecting vertices v_1 and v_4 and another connecting vertices v_2 and v_4. Fig. 1(f) illustrates this for a strip of the square lattice of width $L_y = 4$.

(vii)

$$\mathbf{v_7} = (1, 0, 1, 0, 0)^T \tag{2.29}$$

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which corresponds to a strip with one extra edge connecting vertices v_1 and v_4 and another connecting vertices v_1 and v_3. Fig. 1(g) illustrates this for a strip of the square lattice of width $L_y = 4$. Note that for a strip of the square lattice this end subgraph is equivalent to that described by vector $\mathbf{v_6} = (1, 1, 0, 0, 0)^T$. However, for other types of strips $\mathbf{v_7}$ may represent a different end subgraph.

(viii)

$$\mathbf{v} = (1, 1, 1, 1, 1)^T \tag{2.30}$$

where the end subgraph is the same as the vertical rungs of the strip. Fig. 1(h) illustrates this for a strip of the square lattice of width $L_y = 4$. Fig. 1 illustrates these end subgraphs for a strip of the square lattice of width $L_y = 4$. The different subgraphs are shown on the right end of the strip. Similar subgraphs can be considered on the left end of the strip. In the remainder of this section we consider the end subgraphs of the strips described by the set of vectors above, eqs. (2.23)-(2.30).

(a)
$v_1 = (1,0,0,0,0)$

(b)
$v_2 = (0,1,0,0,0)$

(c)
$v_3 = (0,0,1,0,0)$

(d)
$v_4 = (0,0,0,1,0)$

(e)
$v_5 = (0,0,0,0,1)$

(f)
$v_6 = (1,1,0,0,0)$
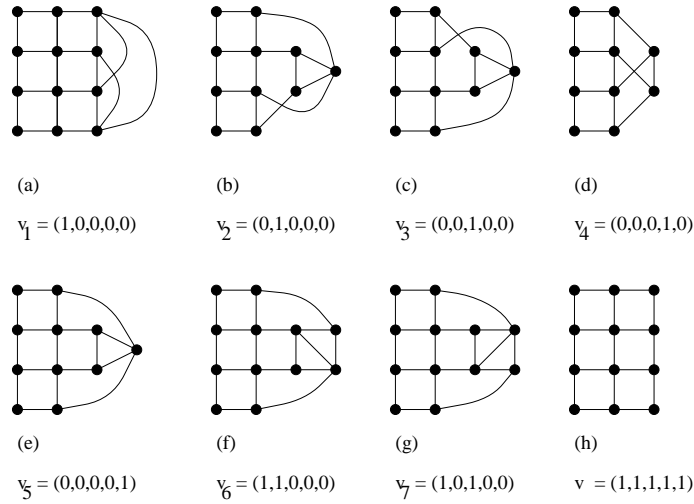
(g)
$v_7 = (1,0,1,0,0)$

(h)
$v = (1,1,1,1,1)$

FIG. 1. Illustration of end subgraphs on a strip of the square lattice of width $L_y = 4$ and free transverse boundary conditions. Cases (a), (b), (c) and (d) contain non-planar end subgraphs, while cases (e), (f), (g) and (h) contain planar end subgraphs.

We label free and periodic boundary conditions in the transverse (y) direction as

$$FBC_y \quad , \quad PBC_y \tag{2.31}$$

respectively, and similarly for the longitudinal (x) direction. The strips have FBC_x and, unless otherwise stated, also FBC_y. For strip graphs with FBC_x and FBC_y, we shall restrict ourselves to cases where the product of matrices MD is a 5×5 matrix. For strip graphs with FBC_x and PBC_y we shall restrict ourselves to cases where the product of matrices MD is a 4×4 matrix. These are strip graphs with cylindrical boundary conditions, with transverse cross sections comprised by four vertices and four edges, forming squares.

Let us denote the four vertices belonging to the transverse cross section of the strip on one of the longitudinal ends of the strip by v_i, with $i = 1, ..., 4$, in sequence. The product of matrices MD is a 4×4 matrix in this case and vectors describing end subgraphs can be

(i)

$$\mathbf{v_1} = (1, 0, 0, 0)^T \tag{2.32}$$

which corresponds to a strip with a K_4 on the end. This K_4 is formed by adding edges connecting vertices v_1 and v_3 and vertices v_2 and v_4.

(ii)

$$\mathbf{v_2} = (0, 1, 0, 0)^T \tag{2.33}$$

corresponding to a strip with vertices v_1 and v_3 contracted and an extra edge connecting vertices v_2 and v_4, yielding a $K_3(v_1 = v_3)$ on the end.

(iii)

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$$\mathbf{v_3} = (0, 0, 1, 0)^T \tag{2.34}$$

which corresponds to a strip with vertices v_2 and v_4 contracted and an extra edge connecting vertices v_1 and v_3, yielding a $K_3(v_2 = v_4)$ on the end.

(iv)

$$\mathbf{v_4} = (0, 0, 0, 1)^T \tag{2.35}$$

which corresponds to a strip with vertices v_2 and v_4 contracted and vertices v_1 and v_3 contracted forming a $K_2(v_1 = v_3, v_2 = v_4)$ on the end.

(v)

$$\mathbf{v_5} = (1, 1, 0, 0)^T \tag{2.36}$$

where the ending subgraph is equal to the repeating unit with an extra edge connecting the non-adjacent pair of vertices v_2 and v_4.

(vi)

$$\mathbf{v_6} = (1, 0, 1, 0)^T \tag{2.37}$$

similar to (v), with the extra edge connecting the other pair of non-adjacent vertices, namely, v_1 and v_3.

(vii)

$$\mathbf{v} = (1, 1, 1, 1)^T \tag{2.38}$$

where the ending subgraph is equal to the vertical rungs along the strip.

In our studies of strip graphs with PBC_y we only consider subgraphs on the two free ends of the strips described by the vectors listed in eqs. (2.32)-(2.38). We next present results for a number of strip graphs of type G'_s.

III. STRIP OF THE SQUARE LATTICE OF WIDTH $L_y = 4$

For a strip of the square (sq) lattice of width $L_y = 4$ vertices and FBC_y, we have (with $f_{sq,i} \equiv f_i$ to save space)

$$
M = \begin{pmatrix}
(q-3)f_0 f_6 & (q-3)f_0 f_1 & (q-3)f_0 f_1 & (q-3)f_0 f_2 & (q-3)f_0 f_1 \\
(q-3)f_0 f_1 & f_0 f_3 & (q-2)f_0 f_2 & f_0 f_5 & f_0 f_4 \\
(q-3)f_0 f_1 & (q-2)f_0 f_2 & f_0 f_3 & f_0 f_5 & f_0 f_4 \\
(q-3)f_0 f_2 & f_0 f_5 & f_0 f_5 & q(q-1)D_4 & (q^2-5q+7)f_0 \\
(q-3)f_0 f_1 & f_0 f_4 & f_0 f_4 & (q^2-5q+7)f_0 & f_0 f_3
\end{pmatrix} \tag{3.1}
$$

where

$$
f_{sq,0} = q(q-1)(q-2) \tag{3.2}
$$

$$
f_{sq,1} = q^3 - 7q^2 + 19q - 20 \tag{3.3}
$$

$$
f_{sq,2} = q^2 - 5q + 8 \tag{3.4}
$$

$$
f_{sq,3} = q^3 - 6q^2 + 14q - 13 \tag{3.5}
$$

$$
f_{sq,4} = q^3 - 7q^2 + 18q - 17 \tag{3.6}
$$

$$
f_{sq,5} = q^2 - 4q + 5 \tag{3.7}
$$

$$
f_{sq,6} = q^4 - 10q^3 + 41q^2 - 84q + 73 \tag{3.8}
$$

Hence, using eq. (2.11) with eqs. (2.7)-(2.9), we obtain the matrix product MD:

$$MD = \begin{pmatrix} f_6 & (q-3)f_1 & (q-3)f_1 & (q-2)(q-3)f_2 & (q-3)f_1 \\ f_1 & f_3 & (q-2)f_2 & (q-2)f_5 & f_4 \\ f_1 & (q-2)f_2 & f_3 & (q-2)f_5 & f_4 \\ f_2 & f_5 & f_5 & D_4 & q^2-5q+7 \\ f_1 & f_4 & f_4 & (q-2)(q^2-5q+7) & f_3 \end{pmatrix} \quad (3.9)$$

Three of the eigenvalues of this MD matrix, which we label as λ_i, $i = 1, 2, 3$ are the inverses of the three roots of the polynomial

$$1 + b_{sq(4),1}x + b_{sq(4),2}x^2 + b_{sq(4),3}x^3 \quad (3.10)$$

where [1]

$$b_{sq(4),1} = -q^4 + 7q^3 - 23q^2 + 41q - 33 \quad (3.11)$$

$$b_{sq(4),2} = 2q^6 - 23q^5 + 116q^4 - 329q^3 + 553q^2 - 517q + 207 \quad (3.12)$$

$$b_{sq(4),3} = -q^8 + 16q^7 - 112q^6 + 449q^5 - 1130q^4 + 1829q^3 - 1858q^2 + 1084q - 279 \quad (3.13)$$

For all end subgraphs studied here, namely, considering any two pairs among (2.23)-(2.30) as the subgraphs on the two ends of the strip, these three eigenvalues contribute to the generating functions of the respective strips. The other two eigenvalues are $\lambda_4 = 1$ and $\lambda_5 = (q^2 - 4q + 3)$. Depending on the end subgraphs chosen, either λ_4 or λ_5 or both contribute to the denominator of the generating function for the chromatic polynomial. However, we find that λ_4 and λ_5 are never leading eigenvalues. Hence, the nonanalytic locus $\mathcal{B}$ of $W(\{G_{sq(4),J,I}\}, q)$ does not depend on the subgraphs J and I on the two ends of the strip and is given by Fig. 3(b) of Ref. [1]. Of course, the actual chromatic polynomials and their zeros for specific finite strips do depend on J and I.

IV. STRIP OF THE HONEYCOMB LATTICE OF WIDTH $L_y = 3$

For a strip of the honeycomb (hc) lattice of width $L_y = 3$ and FBC$_y$, we compute the MD matrix in the same manner as was discussed in detail above for the square lattice. We find that three of the eigenvalues of the MD matrix are the inverses of the roots of

$$1 + b_{hc(3),1}x + b_{hc(3),2}x^2 + b_{hc(3),3}x^3 \tag{4.1}$$

where the coefficients $b_{hc(3),1}$, $b_{hc(3),2}$ and $b_{hc(3),3}$ are given by eqs. (3.14)–(3.16) in Ref. [1]; to save space, we do not list them here. For all end subgraphs studied here, namely, considering any two pairs among (2.23)-(2.30) as the subgraphs on the two ends of the strip, these three eigenvalues contribute to the generating functions of the respective strips. The other two eigenvalues are identically zero. Therefore, just as was the case with the square lattice of width $L_y = 4$, the denominator of the generating function, and thus $\mathcal{B}$, are not modified by the subgraphs on the two ends of the strip.

V. STRIP OF THE TRIANGULAR LATTICE OF WIDTH $L_y = 4$

For a strip of the triangular (t) lattice with width $L_y = 4$, we find that four of these five eigenvalues of the MD matrix are the inverses of the roots of the polynomial

$$1 + b_{t(4),1}x + b_{t(4),2}x^2 + b_{t(4),3}x^3 + b_{t(4),4}x^4 \tag{5.1}$$

where the coefficients $b_{t(4),1}$, $b_{t(4),2}$, $b_{t(4),3}$ and $b_{t(4),4}$ were given respectively by eqs. (B.15)–(B.18) in Ref. [1]. Since their expressions are somewhat lengthy, we do not list them here. Let us refer to these four eigenvalues as λ_i, with $i = 1, 2, 3, 4$. For all end subgraphs studied here, namely, considering any two pairs among (2.23)-(2.30) as the subgraphs on the two ends of the strip, these four eigenvalues contribute to the generating functions of the respective strips. The fifth eigenvalue is $\lambda_5 = 1$, which contributes to the generating function for certain

end subgraphs. λ_5 is leading in a region of the complex q plane and thus modify the curves $\mathcal{B}$ of non-analyticities of $W(\{G_{t(4),J,I}\}, q)$. In Table I we show for various planar and non-planar end subgraphs J and I, whether or not λ_5 contributes to the generating function, and some features of the boundary $\mathcal{B}$ for each case. In turn, the presence or absence of λ_5 in the generating function is determined by the end subgraphs I and J, or equivalently, the vectors $\mathbf{v}_a$ and $\mathbf{v}_b$ via the basic equation (2.21). Indeed, one sees from Table I that $\mathcal{B}$ encloses regions if and only if λ_5 appears in the generating function. This phenomenon, in which $\mathcal{B}$ can depends on the end subgraphs of the strip, is reminiscent of the dependence of certain aspects of complex-temperature phase boundaries and partition function zeros of the Potts model on boundary conditions found in Refs. [24,25].

TABLE I. Illustrative end subgraphs and properties of generating function for a strip of the triangular lattice of width $L_y = 4$. The notation J,I means that the vectors $\mathbf{v_j}$ and $\mathbf{v_i}$ describe the end subgraphs on the two ends of the strip. P stands for planar and NP for non-planar. The third column shows whether or not λ_5 enters in the generating function. The fourth column lists some features of the boundary $\mathcal{B}$.

J,I	planarity	does λ_5 enter?	features of $\mathcal{B}$
$\mathbf{v},\mathbf{v}$	P,P	N	arcs
$\mathbf{v},\mathbf{v_i}$, $i = 1,2,3,4$	P,NP	N	arcs
$\mathbf{v},\mathbf{v_i}$, $i = 5,6,7$	P,P	N	arcs
$\mathbf{v_i},\mathbf{v_i}$, $i = 1,2,3,4$	NP,NP	Y	arcs and one enclosed region
$\mathbf{v_i},\mathbf{v_i}$, $i = 5,6,7$	P,P	N	arcs
$\mathbf{v_1},\mathbf{v_i}$, $i = 2,3,4$	NP,NP	Y	arcs and one enclosed region
$\mathbf{v_1},\mathbf{v_i}$, $i = 5,6,7$	NP,P	N	arcs
$\mathbf{v_6},\mathbf{v_i}$, $i = 1,2,3,4$	P,NP	N	arcs
$\mathbf{v_6},\mathbf{v_i}$, $i = 5,7$	P,P	N	arcs

In Fig. 2(a) we show the analytic structure of the W function for an infinitely long strip of the triangular lattice with width $L_y = 4$ and end subgraphs J, I that do not have λ_5 in the generating function. The resultant nonanalytic locus $\mathcal{B}$ is the same as that which we found in Ref. [1] (see Fig. 5(b) of that paper) for the infinitely long strip of the triangular lattice of the form $(\prod_{\ell=1}^{\infty} H)I$ with width $L_y = 4$, for arbitrary I. The locus $\mathcal{B}$ in Fig. 2(a) consists of arcs that do not separate the complex q plane into separate regions. This locus includes complex-conjugate multiple points of valence three; these are singular points on the algebraic curve constituted by $\mathcal{B}$ in the terminology of algebraic geometry [21]. Similarly, multiple points are observed in many of the loci $\mathcal{B}$ for strip graphs to be described below. Since the locus $\mathcal{B}$ forms by coalescence of zeros of the chromatic polynomial (called chromatic zeros of the graph) as the length of the strip graph goes to infinity, it is of interest to compare the locations of these zeros for a reasonably long finite strip with the asymptotic locus $\mathcal{B}$. Accordingly, we show in this figure the chromatic zeros for a strip with $I = H$ and length $m = 12$.

In Fig. 2(b) we display the analytic structure of the W function for a strip of the triangular lattice with end subgraphs that yield λ_5 in the generating function. In this case the locus $\mathcal{B}$ is comprised by arcs which do enclose a self-conjugate region where λ_5 is the leading eigenvalue. The boundaries of this region cross the real q axis at $q = q_1 = \frac{3+\sqrt{5}}{2} = 2.618...$ and at $q = 3$. Following our earlier work [11,12], we define q_c as the maximal finite real value of q where W is nonanalytic for a given lattice, i.e., where $\mathcal{B}$ crosses the real axis. Hence, for the present strip, $q_c = 3$. We note that for the infinite triangular lattice, $q_c = 4$ [12]. For comparison, in 2(b) we exhibit the zeros of the chromatic polynomial $P((G_{t(4),J,I})_m, q)$ with subgraphs on the right and left ends of the strip described by $J = I = \mathbf{v}_2$ and $m = 10$ repeating units in the strip.

We can understand this difference in the topology of the loci $\mathcal{B}$ as follows. One complex-conjugate pair of arc endpoints in the locus $\mathcal{B}$ in Fig. 2(a) occurs at $q \simeq 2.759 \pm 0.154i$.

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However, for the triangular $L_y = 4$ strips with endgraphs that yield λ_5 in the generating function, this pair of would-be endpoints, and the contiguous portion of the associated arcs, do not actually occur on $\mathcal{B}$ because they lie inside the enclosed region in Fig. 2(b) where λ_5 is leading. Thus, as was anticipated in our discussion in Ref. [1], the difference in topology of $\mathcal{B}$ for the $(\prod_{\ell=1}^{\infty} H)I$ strips and the present $J(\prod_{\ell=1}^{\infty} H)I$ strip of the triangular lattice is associated with the fact that for the former types of strips, the leading λ's were always algebraic, whereas here a leading λ, viz., $\lambda_5 = 1$ is a (constant) polynomial, rather than an algebraic function of q. Here we see how these features that $\mathcal{B}$ can (i) depend on J, I and (ii) enclose regions are both connected with each other via the basic eq. (2.21), which determines which λ's appear in the generating function and, in particular, whether the leading λ's are always algebraic, as for $(\prod_{\ell=1}^{\infty} H)I$ strips with nontrivial $\mathcal{B}$ loci, or can also include polynomial λ's, as for $J(\prod_{\ell=1}^{\infty} H)I$ strips.
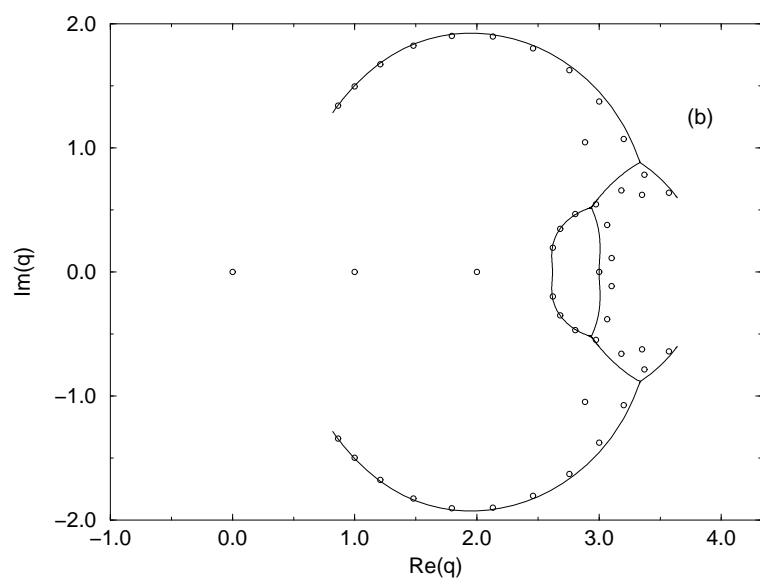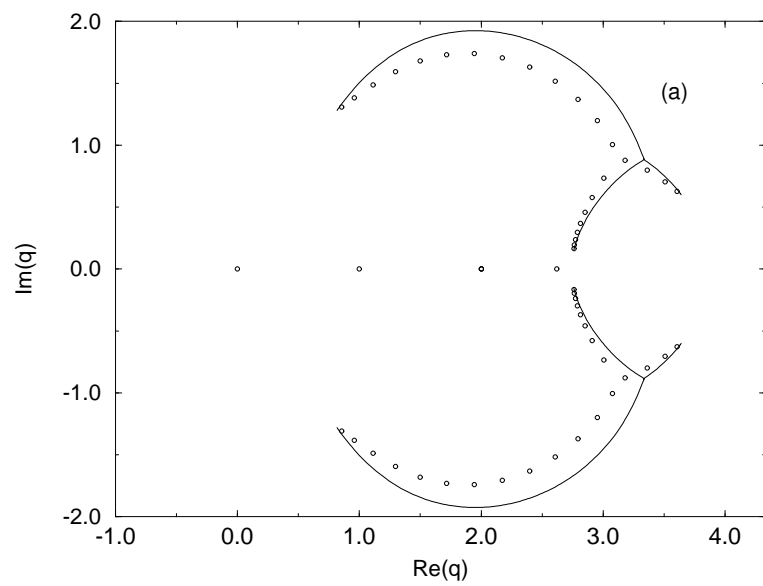
FIG. 2. Analytic structure of the function $W(\{G_{t(4),J,I}\}, q)$ for a (a) $(L_x = \infty) \times (L_y = 4)$ strip of the triangular lattice given by $\lim_{m\to\infty} J(\prod_{\ell=1}^m H)I$ with end graphs I, J that do not yield λ_5 in the generating function (this includes the case where J is absent); (b) $\lim_{m\to\infty} J(\prod_{\ell=1}^m H)I$ with end subgraphs I, J that yield λ_5 in the generating function. For comparison, the zeros of the chromatic polynomials are shown for the strips with (a) $I = H$ and $m = 12$ (whence $n = 56$); (b) subgraphs $I = J$ described by $\mathbf{v_2}$ and $m = 10$ (whence $n = 46$).

Thus, with changes in the subgraphs on the two ends I and J of a strip of the triangular lattice of width $L_y = 4$, the analytic structure of $W(\{G_{t(4),J,I}\}, q)$ changes from a set of arcs to structures which enclose regions. For this type of strip we only obtained enclosed regions when the subgraphs on both ends of the strip were non-planar. However, in Ref. [19] an example is given where this happens with planar end subgraphs, albeit with a heteropolygonal Archimedean lattice, viz., the $(3^3 \cdot 4^2)$ lattice. Our present calculation shows that for the strip graphs of the form $J(\prod_{\ell=1}^\infty H)I$, the nonanalytic locus $\mathcal{B}$ may depend on the end subgraphs I and J, in contrast to the property proved in Ref. [1] that for strip graphs of the form $(\prod_{\ell=1}^\infty H)I$, the resultant $\mathcal{B}$ is independent of the end subgraph I.

VI. STRIP OF THE $(4 \cdot 8^2)$ LATTICE OF WIDTH $L_y = 3$

We have also studied strips of lattices that are tilings involving more than one type of regular polygon. We recall [22,23,16,1] that an Archimedean lattice is a uniform tiling of the plane by one or more regular polygons, such that each vertex is equivalent to every other vertex. An Archimedean lattice is identified by the symbol

$$\Lambda = (\prod_i p_i^{a_i}) \qquad (6.1)$$

where the product is the ordered sequence of polygons that one traverses in making a complete circuit around a vertex in a given (say counterclockwise) direction, and the notation

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$p_i^{a_i}$ indicates that the regular p-sided polygon p_i occurs contiguously a_i times in this circuit. In this section we consider a strip of the $(4 \cdot 8^2)$ lattice of width $L_y = 3$ with free boundary conditions in the y direction. For this strip, the MD matrix is 5×5, and three of the eigenvalues of this matrix are the inverses of the roots of $1 + b_{488(3),1}x + b_{488(3),2}x^2 + b_{488(3),3}x^3$, where the coefficients $b_{488(3),j}$, $j = 1, 2, 3$ were given in Ref. [1]. For all of the end graphs I and J studied here, namely considering any two pairs among (i)-(viii) in eqs. (2.23)–(2.30), these three eigenvalues contribute to the generating functions of the respective strips. The other two eigenvalues are identically zero. Therefore, as was the case with the strip of the honeycomb lattice with $L_y = 3$, the denominator of the generating function and thus $\mathcal{B}$, are not modified by the end graphs I and J on the strip.

VII. STRIP OF THE SQUARE LATTICE WITH PBC$_y$

In this section we consider a strip of square lattice with PBC$_y$, i.e., periodic boundary conditions in the transverse direction ($=$ cylindrical boundary conditions). We begin by observing that if one takes a strip of a regular lattice with FBC$_y$ and identifies the vertices with lowest and highest values of y for each x to produce PBC$_y$, the changes in $\mathcal{B}$ depend on the specific type of lattice and value of L_y. For example, let us consider the square lattice with FBC$_y$ and $L_y = 4$ and carry out the above-mentioned operation, equivalent to gluing together the upper and lower edges of the strip, so that the transverse cross sections are triangles. Although the infinite-length limit of the original strip with FBC$_y$ yielded a nontrivial $\mathcal{B}$, shown in Fig. 3(b) of Ref. [1], after the gluing operation, the chromatic polynomial factorizes trivially for the square strip with PBC$_y$ and triangular cross section, and $\mathcal{B} = \emptyset$, i.e. there is no locus of points where W is nonanalytic in the q plane. In contrast, for a number of cases to be discussed below, the gluing operation takes an infinite strip with nontrivial $\mathcal{B}$ to another infinite strip with nontrivial and different $\mathcal{B}$. As we shall

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show below, for strips of the square (triangular) lattice with transverse cross sections forming squares, and end graphs I and J that are the same as the other vertical rungs of the strip, the resultant $\mathcal{B}$ does not (does) enclose regions, respectively.

We begin by studying the strip of the square lattice with PBC$_y$ and transverse cross sections forming squares. Depending on one's labelling conventions, this corresponds to $L_y = 4$ or $L_y = 5$, where in the latter case, one interprets the periodic boundary conditions as identifying the top and bottom vertices for each value of x. It is interesting to observe that this graph can be regarded as part of a three dimensional simple cubic lattice. For end subgraphs I and J described by the vector $\mathbf{v}$ we obtain a generating function with $j_{max} = 1$, $k_{max} = 2$, and coefficients

$$a_{sq(4),c,0} = q(q-1)(q^6 - 11q^5 + 55q^4 - 159q^3 + 282q^2 - 290q + 133) \tag{7.1}$$

$$a_{sq(4),c,1} = -q(q-1)(q^2 - 3q + 3)(q^6 - 12q^5 + 61q^4 - 169q^3 + 269q^2 - 231q + 85) \tag{7.2}$$

$$b_{sq(4),c,1} = -q^4 + 8q^3 - 29q^2 + 55q - 46 \tag{7.3}$$

$$b_{sq(4),c,2} = q^6 - 12q^5 + 61q^4 - 169q^3 + 269q^2 - 231q + 85 \tag{7.4}$$

In this case the MD matrix is the upper left-hand 4×4 submatrix of MD given above in eq. (3.9) for the square strip with FBC$_y$. Two of the four eigenvalues of the MD matrix are the inverses of the roots of $1 + b_{sq(4),c,1}x + b_{sq(4),c,2}x^2$. These two eigenvalues, denoted λ_1 and λ_2, contribute to the generating functions for all end graphs J and I that we consider. The other two eigenvalues are $\lambda_3 = 1$ and $\lambda_4 = (q-1)(q-3)$, which contribute to the generating functions for some end graphs. In Table II we show for various end subgraphs J, I whether or not λ_3 and λ_4 contribute to the generating function and some features of the boundaries $\mathcal{B}$ for each case.

TABLE II. Illustrative end subgraphs and properties of generating function for a strip of the square lattice with periodic boundary conditions on the transverse direction and transverse cross sections forming squares. Notation is as in Table I. The second and third columns indicate whether or not λ_3 and λ_4, respectively, contribute to the generating function.

| J, I	does λ_3 enter?	does λ_4 enter?	features of $\mathcal{B}$
$\mathbf{v}, \mathbf{v}$	N	N	arcs
$\mathbf{v_i}, \mathbf{v_i}$, $i = 1, 4$	Y	N	arcs
$\mathbf{v_i}, \mathbf{v_i}$, $i = 2, 3$	Y	Y	arcs and one pair of enclosed regions
$\mathbf{v_i}/\mathbf{v_i}$, $i = 5, 6$	N	Y	arcs and one pair of enclosed regions
$\mathbf{v}, \mathbf{v_i}$, $i = 1, .., 6$	N	N	arcs
$\mathbf{v_1}, \mathbf{v_i}$, $i = 2, 3, 4$	Y	N	arcs
$\mathbf{v_1}, \mathbf{v_i}$, $i = 5, 6$	N	N	arcs
$\mathbf{v_2}, \mathbf{v_3}$	Y	Y	arcs and one pair of enclosed regions
$\mathbf{v_2}, \mathbf{v_i}$, $i = 5, 6$	N	Y	arcs and one pair of enclosed regions
$\mathbf{v_3}, \mathbf{v_i}$, $i = 5, 6$	N	Y	arcs and one pair of enclosed regions
$\mathbf{v_4}, \mathbf{v_i}$, $i = 2, 3$	Y	N	arcs
$\mathbf{v_4}, \mathbf{v_i}$, $i = 5, 6$	N	N	arcs
$\mathbf{v_5}, \mathbf{v_6}$	N	Y	arcs and one pair of enclosed regions

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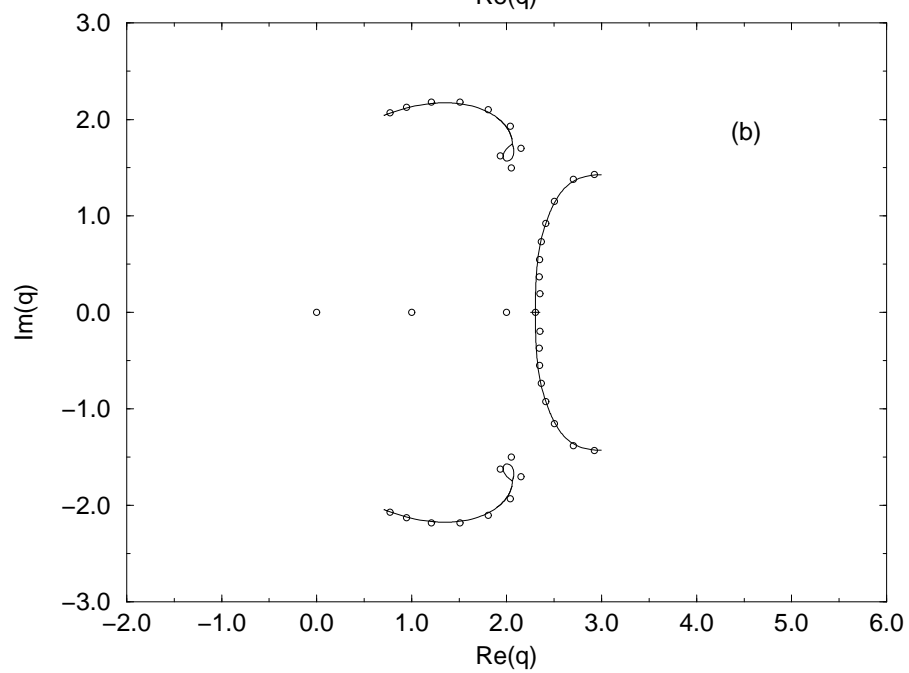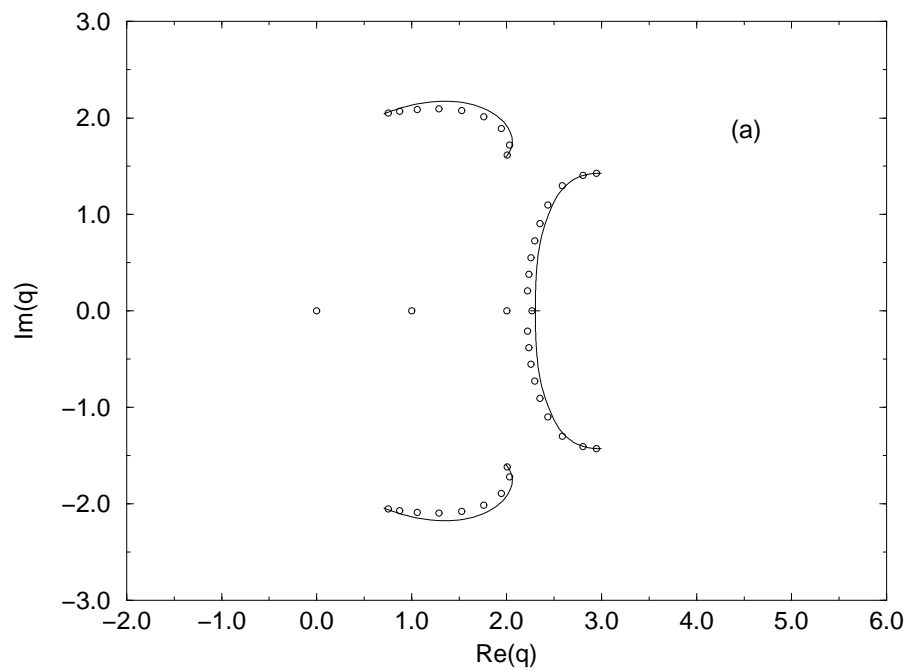
FIG. 3. Analytic structure of the function W for an infinitely long strip of square lattice with periodic boundary condition on the transverse direction and transverse cross sections forming squares: (a) corresponds to end graphs I and J for which λ_4 does not enter in the generating function. (b) corresponds to cases where λ_4 enters in the generating function. For comparison, the zeros of the chromatic polynomial for a strip with $m = 8$ repeating units ($n = 40$ vertices in (a) and $n = 36$ vertices in (b)) and end graphs I and J described by (a) $\mathbf{v}$ [(b) $\mathbf{v_2}$] on the right and left ends of the strip, are shown.

We note that although λ_3 enters in the generating functions for some end graphs, it is never leading, so the analytic structure is not modified by its presence in the generating function. For end graphs where λ_4 is not present in the generating function the nonanalytic locus $\mathcal{B}$ of the W function, shown in Fig. 3(a), includes one complex-conjugate pair of arcs and one self-conjugate arc which crossing the real q axis at $q \simeq 2.3026$. There is also a line segment on the real q axis from $q = 2.2534$ to $q = 2.3517$, which latter point is thus the value of q_c for (the $m \to \infty$ limit of) this strip. The locus $\mathcal{B}$ in Fig. 3(a) is somewhat similar to that which we presented in Fig. 3(a) of Ref. [1] for the infinitely long strip of the square lattice of width $L_y = 3$ with FBC$_y$ except that in that case, the self-conjugate arc crossed the real q axis at $q = 2$, which was thus the value of q_c in that case. The value of q_c that we find for PBC$_y$, viz., $q_c \simeq 2.35$, is comparable to the value $q_c \simeq 2.27$ that we found for the infinitely long strip of the square lattice with FBC$_y$ (see Fig. 3(b) of Ref. [1]) and width $L_y = 4$. The endpoints of the arcs and the line segment correspond to the branch points of the eigenvalues λ_1 and λ_2. For end graphs I and J where λ_4 is present in the generating function the analytic structure of the W function, shown in Fig. 3(b), has an extra pair of complex-conjugate regions. In these enclosed regions the eigenvalue λ_4 is leading. The complex-conjugate pair of branch points of the eigenvalues λ_1 and λ_2 at $q = 1.992 \pm 1.594i$ lie within the complex-conjugate enclosed regions, where λ_4 is leading. Hence, these branch points do not correspond to arc endpoints.

VIII. STRIP OF THE TRIANGULAR LATTICE WITH PBC$_y$

We next consider a strip of the triangular lattice with PBC$_y$ and transverse cross sections forming squares. As above, depending on one's labelling conventions, this corresponds to $L_y = 4$ or $L_y = 5$, where in the latter case one identifies the top and bottom vertices for each value of x. The subgraphs on the horizontal ends of the strip are described by the vector $\mathbf{v}$. We obtain a generating function with $j_{max} = 1$, $k_{max} = 2$, and coefficients

$$a_{t(4),c,0} = q(q-1)(q-2)(q-3)(q^4 - 10q^3 + 41q^2 - 84q + 71) \tag{8.1}$$

$$a_{t(4),c,1} = -2q(q-1)(q-2)(q-3)^3(q^2 - 5q + 5)(q^2 - 3q + 3) \tag{8.2}$$

$$b_{t(4),c,1} = -(q-3)(q^3 - 9q^2 + 33q - 48) \tag{8.3}$$

$$b_{t(4),c,2} = 2(q-2)(q-3)^3(q^2 - 5q + 5) \tag{8.4}$$

Two of the four eigenvalues of the MD matrix are the inverses of the roots of $1 + b_{t(4),c,1}x + b_{t(4),c,2}x^2$. These two eigenvalues, denoted λ_1 and λ_2 for the $+$ and $-$ signs of the square root, respectively, contribute to the generating functions for all end graphs I and J that we consider. The other two eigenvalues are $\lambda_3 = 0$ and $\lambda_4 = 2$, the latter contributes to the generating functions for some end graphs. In Table III we show for various subgraphs J, I on the two ends of the strips, whether or not λ_4 contributes to the generating function and some features of the boundary $\mathcal{B}$ for each case.

TABLE III.   Illustrative end subgraphs and properties of generating function for a strip of the triangular lattice with periodic boundary conditions on the transverse direction and transverse cross sections forming squares. The notation is as in Table I. The second column indicates whether or not λ_4 contributes to the generating function.

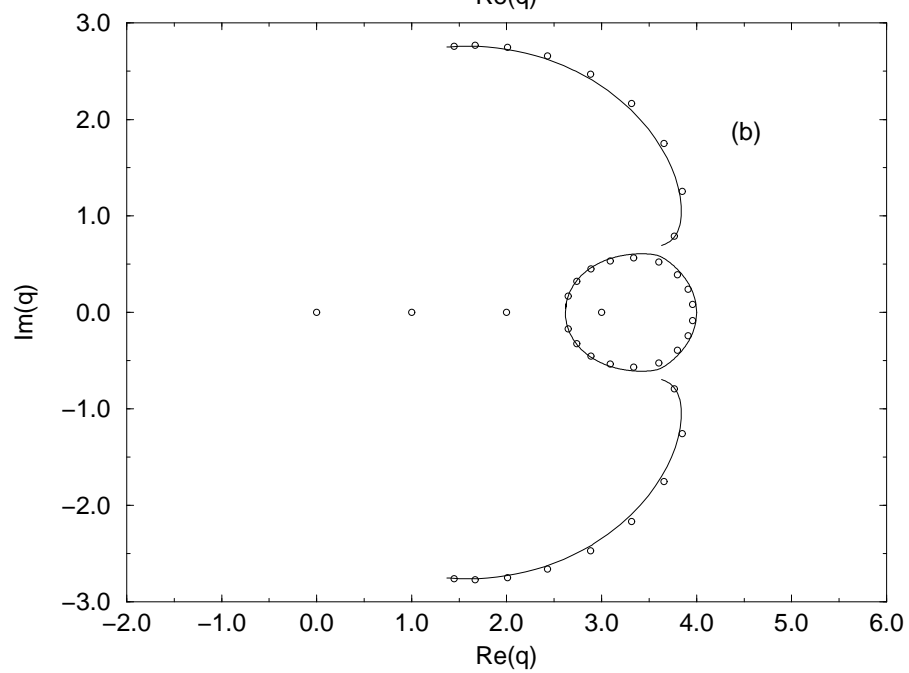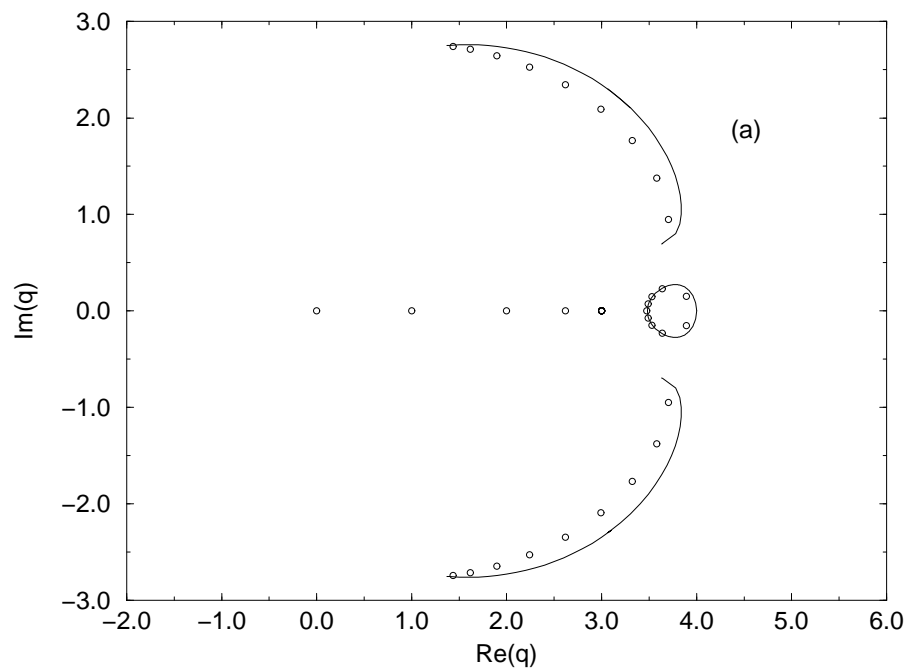| boundaries	does λ_4 enter?	features of $\mathcal{B}$
$\mathbf{v}/\mathbf{v}$	N	arcs and closed region (a)
$\mathbf{v}/\mathbf{v_i}$, $i = 1, .., 6$	N	arcs and closed region (a)
$\mathbf{v_1}/\mathbf{v_i}$, $i = 1, .., 6$	Y	arcs and closed region (b)
$\mathbf{v_2}/\mathbf{v_i}$, $i = 2, .., 6$	Y	arcs and closed region (b)
$\mathbf{v_3}/\mathbf{v_i}$, $i = 3, .., 6$	Y	arcs and closed region (b)
$\mathbf{v_4}/\mathbf{v_i}$, $i = 4, .., 6$	Y	arcs and closed region (b)
$\mathbf{v_5}/\mathbf{v_i}$, $i = 5, 6$	N	arcs and closed region (a)
$\mathbf{v_6}/\mathbf{v_6}$	N	arcs and closed region (a)

FIG. 4. Analytic structure of the function W for the strip of triangular lattice with periodic boundary condition on the transverse direction, transverse cross sections forming squares, and length $L_x = \infty$. (a) corresponds to end graphs I and J for which λ_4 does not enter in the generating function. (b) corresponds to cases where λ_4 enters in the generating function. For comparison, the zeros of the chromatic polynomial for a strip with $m = 8$ repeating units ($n = 40$ vertices) and end graphs I and J described by (a) $\mathbf{v}$ [(b) $\mathbf{v_1}$] on the right and left ends of the strip, are shown.

For end graphs where λ_4 is not present in the generating function the analytic structure of the W function, shown in Fig. 4(a), is formed by one pair of complex-conjugate arcs and one self-conjugate region. Outside (inside) the enclosed region λ_1 (λ_2) is leading, The endpoints of the arcs correspond to the branch points of the eigenvalues λ_1 and λ_2. The boundary $\mathcal{B}$ crosses the real q axis at $q \simeq 3.481$ and at $q = 4$. For end graphs where λ_4 is present in the generating function the analytic structure of the W function, shown in Fig. 4(b), is qualitatively the same as the previous case, except that the enclosed region is larger and λ_4 is leading therein. The boundary $\mathcal{B}$ crosses the real q axis at $q = \frac{3+\sqrt{5}}{2} = 2.618...$ and at $q = 4$. It is interesting to note that for a strip of the triangular lattice with PBC_y and transverse cross sections forming squares and different end graphs I and J, our results yield a value of q_c which coincides with the value for an infinite triangular lattice, namely, $q_c = 4$ [12]. However, this is not a general result; in our analysis above of the strip of the square lattice with PBC_y and transverse cross sections forming squares, we obtained $q_c \simeq 2.35$, which is somewhat less than the value $q_c = 3$ for the infinite square lattice.

IX. DISCUSSION AND CONCLUSIONS

In this paper we have presented exact calculations of chromatic polynomials P and asymptotic limiting W functions for strip graphs of the form $J(\prod_{\ell=1}^{\infty} H)I$, where J and I are various subgraphs on the left and right ends of the strip, and ($\prod_{\ell=1}^{m} H$) are strips of various regular

lattices consisting of m-fold repetitions of subgraph units H. We have also studied the effects of using periodic as well as free boundary conditions in the transverse direction. For the respective strip graphs we have determined the loci $\mathcal{B}$ where W is nonanalytic in the complex q plane. Our present analysis generalizes our earlier calculations for strip graphs of the form $(\prod_{j=1}^{\infty} H)I$.

In contrast to the $(\prod_{\ell=1}^{\infty} H)I$ strip graphs, where $\mathcal{B}$ (i) is independent of I, and (ii) consists of arcs and possible line segments that do not enclose any regions in the q plane, we find that for some $J(\prod_{\ell=1}^{\infty} H)I$ strip graphs, $\mathcal{B}$ (i) does depend on I and J, and (ii) can enclose regions in the q plane. We have explained these differences in the present work and have related them, via eq. (2.21) to whether the leading λ's are algebraic, as in the case of $(\prod_{\ell=1}^{\infty} H)I$ strips with nontrivial $\mathcal{B}$, or can also include polynomial λ's, as in certain of the $J(\prod_{\ell=1}^{\infty} H)I$ strips. It should be noted that even if all of the leading λ's are algebraic, the resultant locus $\mathcal{B}$ can still enclose regions. This can occur when this locus $\mathcal{B}$ is noncompact in the q plane, extending infinitely far from the origin [14]. Indeed, we showed in Ref. [14] that whenever $\mathcal{B}$ is noncompact in the q plane, passing through $1/q = 0$, it encloses regions. An example of a family of graphs with only algebraic λ's which has a locus $\mathcal{B}$ that encloses regions (and is noncompact) is the $\{U_k\}$ family discussed in detail in (the third paper of) Ref. [14]. Our present study thus elucidates the effects of different end subgraphs I and J and of free versus periodic transverse boundary conditions on the function W describing the ground state degeneracy, per site, of the Potts antiferromagnet on infinitely long 2D strips of various lattices.

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